< 2.3. Models of the fluid >

❖ Physical laws

- In developing the equations of aerodynamics we will invoke the firmly established and time-tested physical laws:
  - Conservation of mass
  - Conservation of momentum
  - Conservation of energy

< 2.3. Models of the fluid >

❖ Physical laws

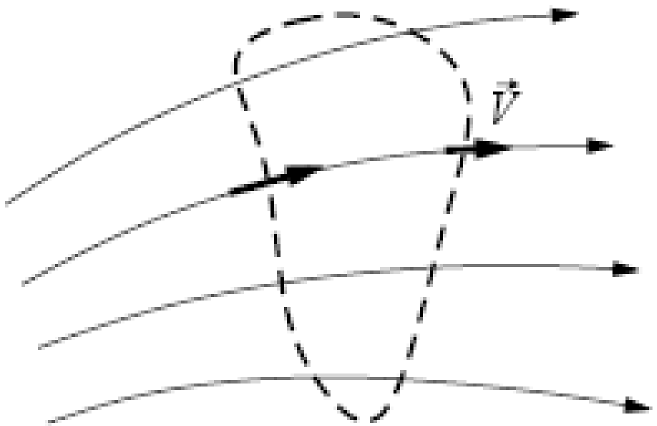
- Because we are not dealing with isolated point masses, but rather a continuous deformable medium, we will require new conceptual and mathematical techniques to apply these laws correctly.

  - Finite control volume approach
  - Infinitesimal fluid element approach
  - Molecular approach

< 2.3. Models of the fluid >

❖ Control volume approach

- One concept is the <u>*control volume*</u>, which can be either finite or infinitesimal. Two types of control volumes can be employed :
  - Volume is fixed in space (Eulerian type). Fluid can freely pass through the volume's boundary.

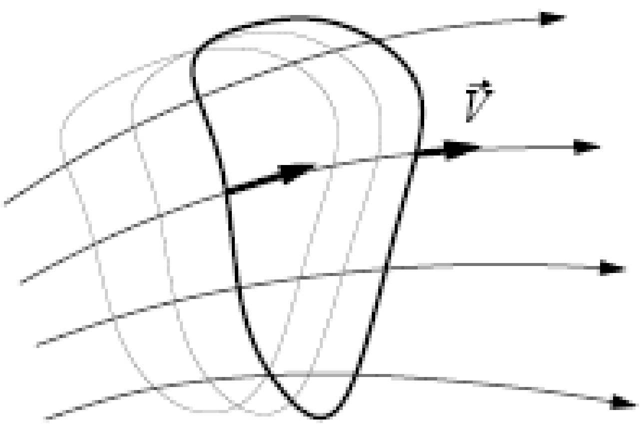


Eulerian Control Volume
fixed in space

< 2.3. Models of the fluid >

Control volume approach

- One concept is the *control volume*, which can be either finite or infinitesimal. Two types of control volumes can be employed : (cont'd)
  - Volume is attached to the fluid (Lagrangian type). Volume is freely carried along with the fluid, and no fluid passes through its boundary. This is essentially the same as the free-body concept employed in solid mechanics.

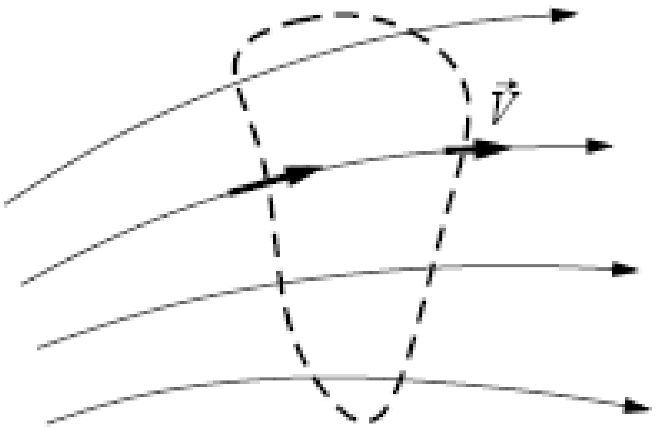


Lagrangian Control Volume moving with fluid

< 2.3. Models of the fluid >

❖ Control volume approach

- One concept is the *control volume*, which can be either finite or infinitesimal. Two types of control volumes can be employed : (cont'd)
  - Both approaches are valid.
    Here we will focus on the *fixed control volume*. (Eulerian type)

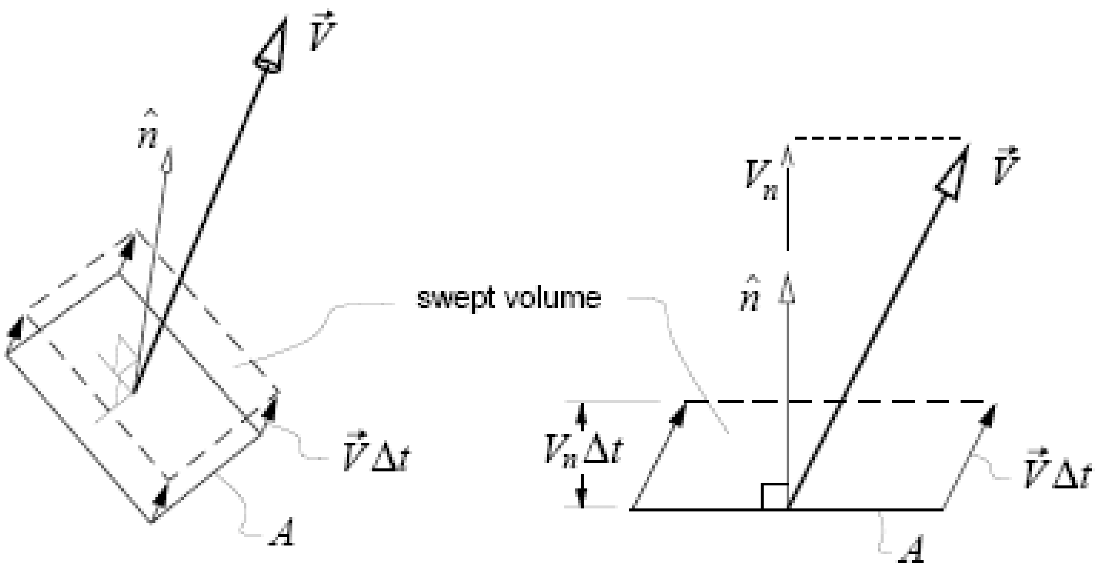


Eulerian Control Volume fixed in space

< 2.4. Continuity equation >

❖ Mass flow

- Consider a small patch of the surface of the fixed, permeable control volume. The patch has area *A*, and normal unit vector ***n***.

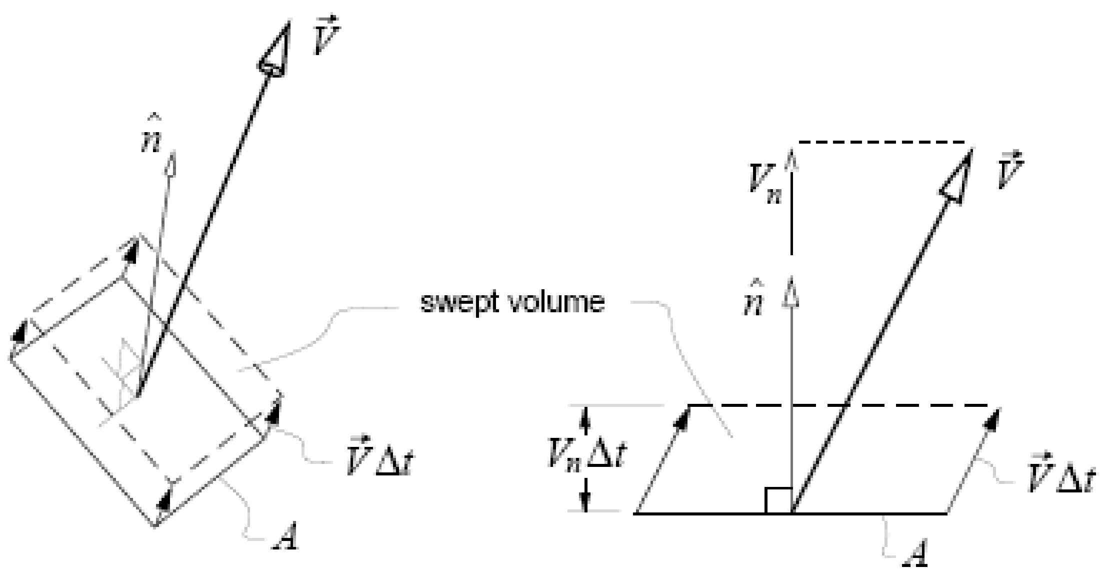


< 2.4. Continuity equation >

❖ Mass flow

- The plane of fluid particles which are on the surface at time t will move off the surface at time $t+\Delta t$, sweeping out a volume given by $\Delta v = V_n A \Delta t$.

(where $V_n = \mathbf{V} \cdot \mathbf{n}$)

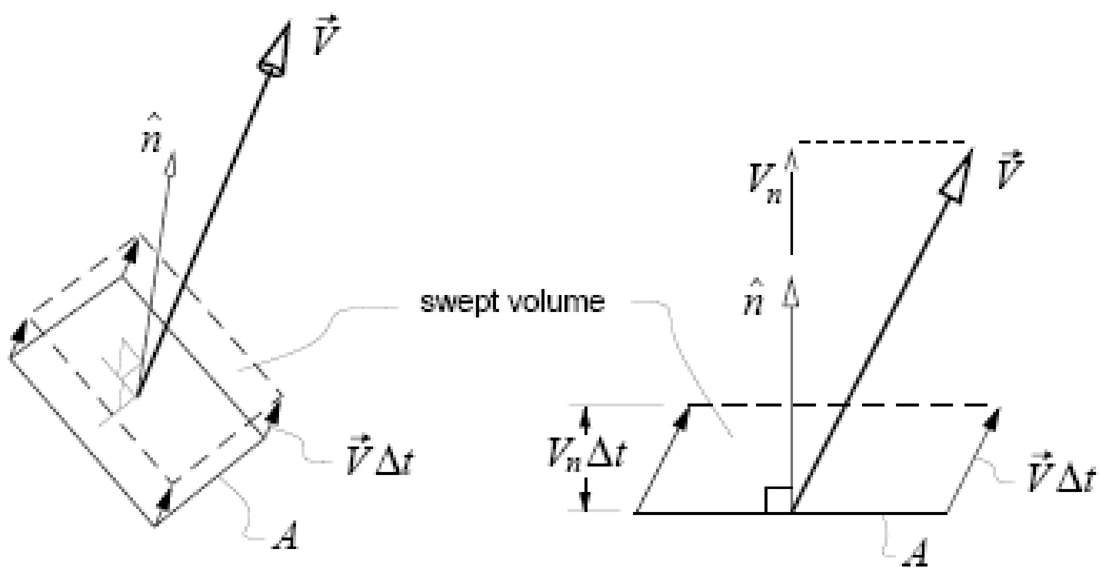


< 2.4. Continuity equation >

❖ Mass flow

- The mass of fluid in this swept volume, which evidently passed through the area during the *Δt* interval, is

$$\Delta m = \rho \Delta v = \rho V_n A \Delta t$$



< 2.4. Continuity equation >

Mass flow

- The *mass flow* is defined as the time rate of this mass passing though the area.

$$mass\ flow = \dot{m} = \lim_{\Delta t \to 0} \frac{\Delta m}{\Delta t} = \rho V_n A$$

- The *mass flux* is defined simply as mass flow per area.

$$mass\ flux = \frac{\dot{m}}{A} = \rho V_n$$

< 2.4. Continuity equation >

❖ Mass conservation application

- The conservation of mass principle can now be applied to the finite fixed control volume, but now it must allow for the possibility of mass flow across the volume boundary.

$$\frac{d}{dt}\left(Mass\ in\ volume\right) = Mass\ \ flow\ in\,to\ volume$$

< 2.4. Continuity equation >

Mass conservation application

- Using the previous relations we have

$$\frac{d}{dt}\iiint \rho dv = -\oiint \rho \vec{V} \cdot \hat{n} dA$$

where the negative sign is necessary because ***n*** is defined to point outwards, so an inflow is where -***V***·***n*** is positive.

< 2.4. Continuity equation >

❖ Mass conservation application

- Using Gauss's Theorem and bringing the time derivative inside the integral we have the result

$$\iiint \left[\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho \vec{V} \right) \right] dv = 0$$

< 2.4. Continuity equation >

❖ Mass conservation application

- This relation must hold for any control volume whatsoever. If we place an infinitesimal control volume at every point in the flow and apply the above equation, we can see that the whole quantity in the brackets must be zero at every point. This results in the *Continuity Equation*

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho \vec{V} \right) = 0$$

which is the embodiment of the Mass Conservation principle for fluid flow.

< 2.4. Continuity equation >

❖ Mass conservation application

- The *steady flow* version is

$$\nabla \cdot \left(\rho \vec{V} \right) = 0$$

- For *low-speed flow*, steady or unsteady, the density ρ is essentially constant, which gives the very great simplification that the velocity vector field has zero divergence.

$$\nabla \cdot \vec{V} = 0$$

< 2.4. Continuity equation >

❖ Mass conservation application

- All of the above forms of the continuity equation are used in practice. The *surface-integral form* with the *steady assumption*,

$$\oiint \rho \vec{V} \cdot \hat{n} dA = 0$$

is particularly useful in many engineering applications.

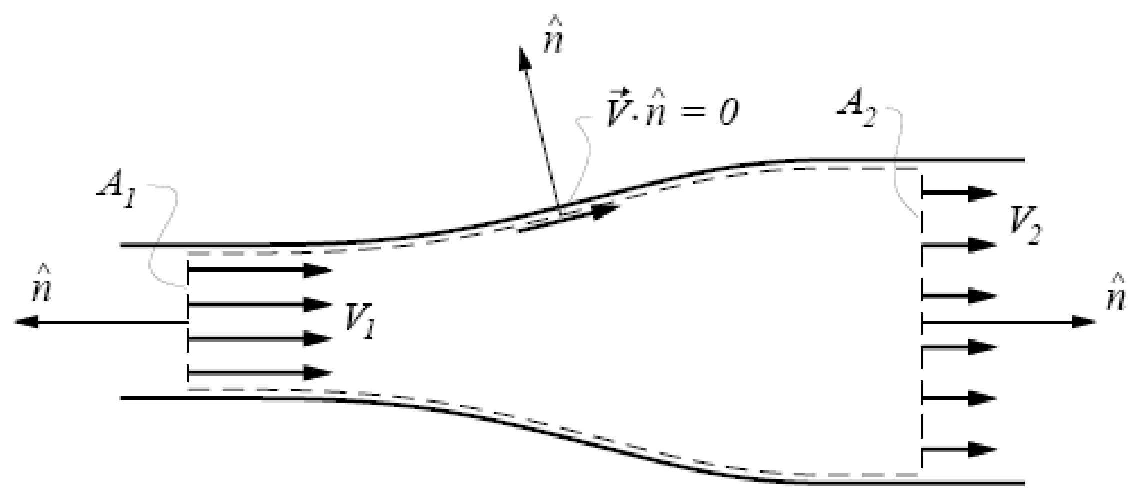
< 2.4. Continuity equation >

❖ Channel flow application

- Placing the control volume inside a pipe or channel of slowly-varying area, we now evaluate above equation.

$$\oiint \rho \vec{V} \cdot \hat{n} dA = -\rho_1 V_1 A_1 + \rho_2 V_2 A_2 = 0$$

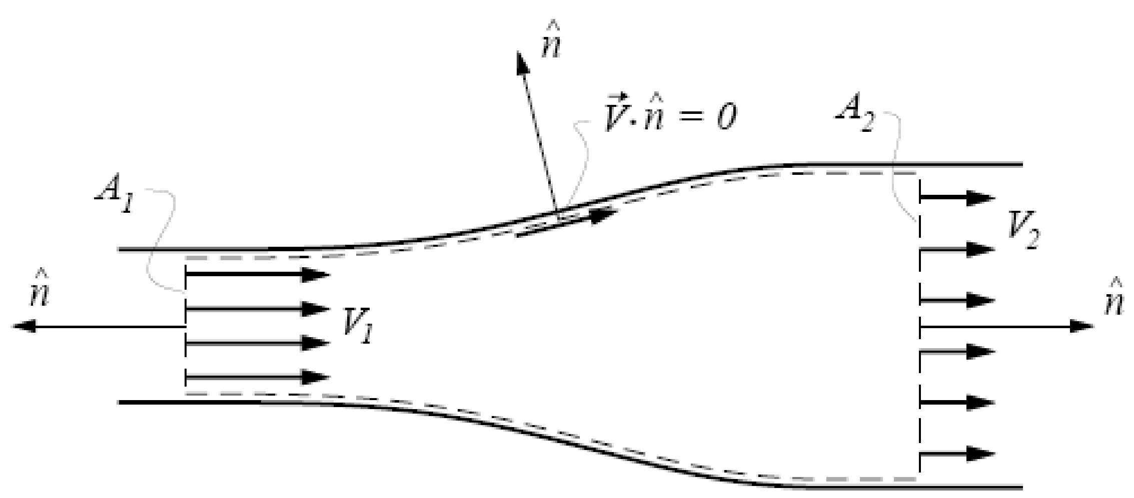
$$\rho_1 V_1 A_1 = \rho_2 V_2 A_2$$



< 2.4. Continuity equation >

❖ Channel flow application

- The negative sign for station 1 is due to $\boldsymbol{V}\cdot\boldsymbol{n}=-V_1$ at that location. The control volume faces adjacent to walls do not contribute to the integral, since their normal vectors are perpendicular to the local flow and therefore have $\boldsymbol{V}\cdot\boldsymbol{n}=0$.



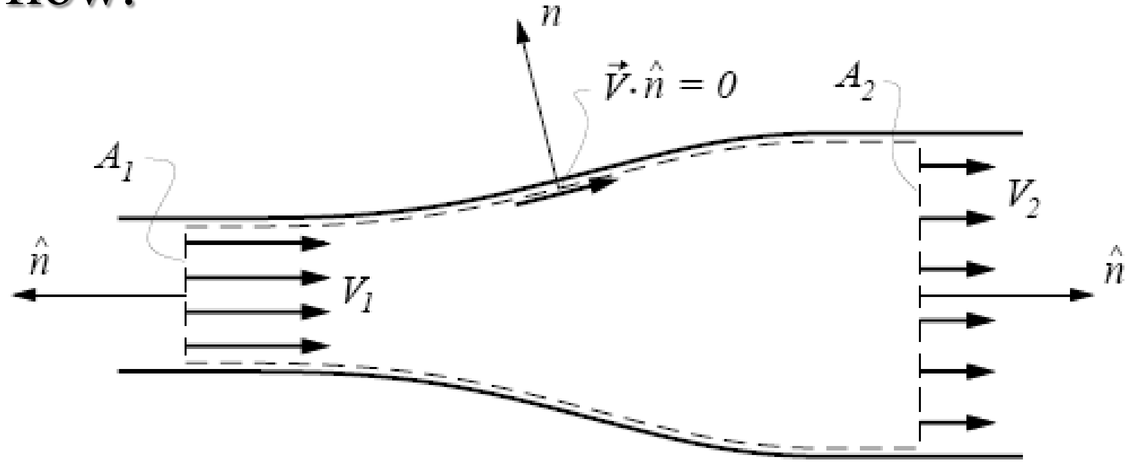
< 2.4. Continuity equation >

Channel flow application

- Placing plane 2, say, at any other location, gives the general result that

$$\rho VA = const.$$

The product ρVA is also recognized as the constant channel mass flow.

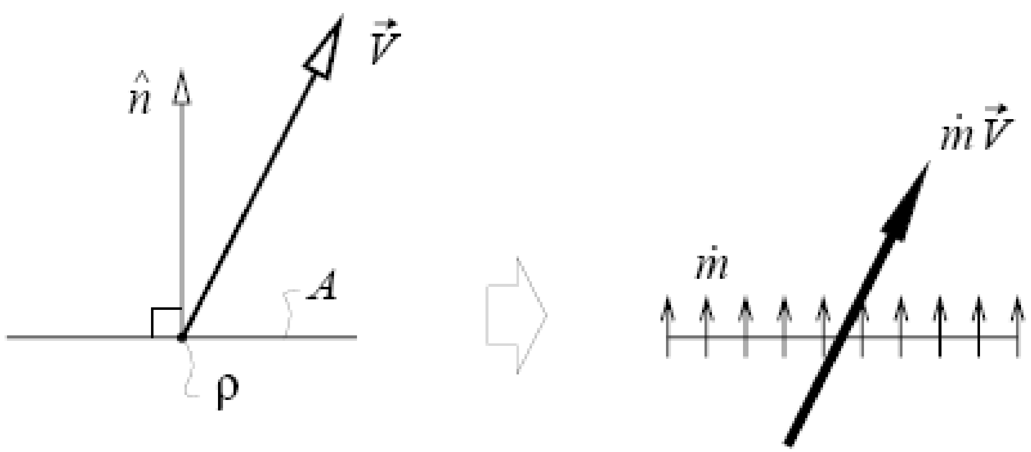


< 2.5. Momentum equations >

❖ Momentum flow

- When material flows through the surface, it carries not only mass, but momentum as well. The momentum flow can be described as

$$\overrightarrow{momentum}\ \ flow = \left(mass\ \ flow \right) \times \left(\overrightarrow{momentum} / \, mass \right)$$



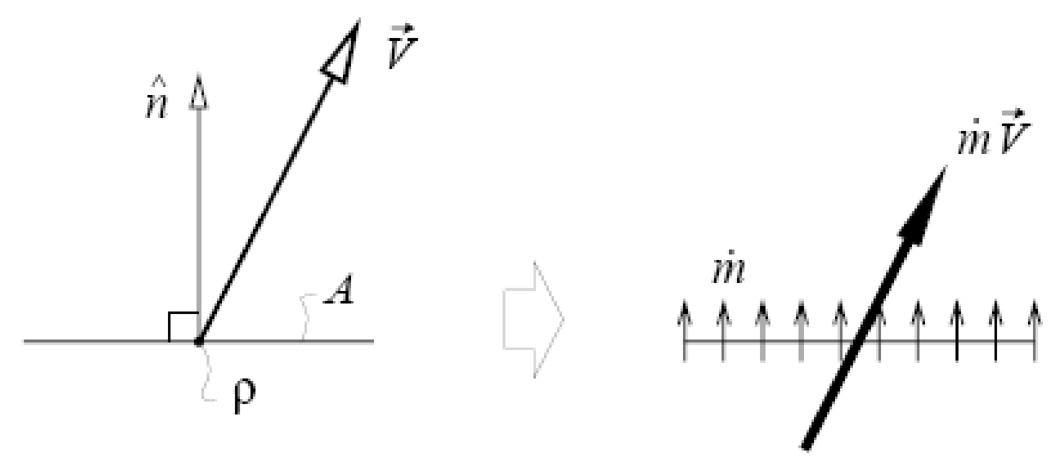
< 2.5. Momentum equations >

❖ Momentum flow

- where the mass flow was defined earlier, and the momentum/mass is simply the velocity vector ***V***. Therefore

$$\overrightarrow{momentum}\ flow = \dot{m}\vec{V} = \rho\left(\vec{V}\cdot\hat{n}\right)A\vec{V} = \rho V_n A\vec{V}$$

where V_n=***V·n*** as before.

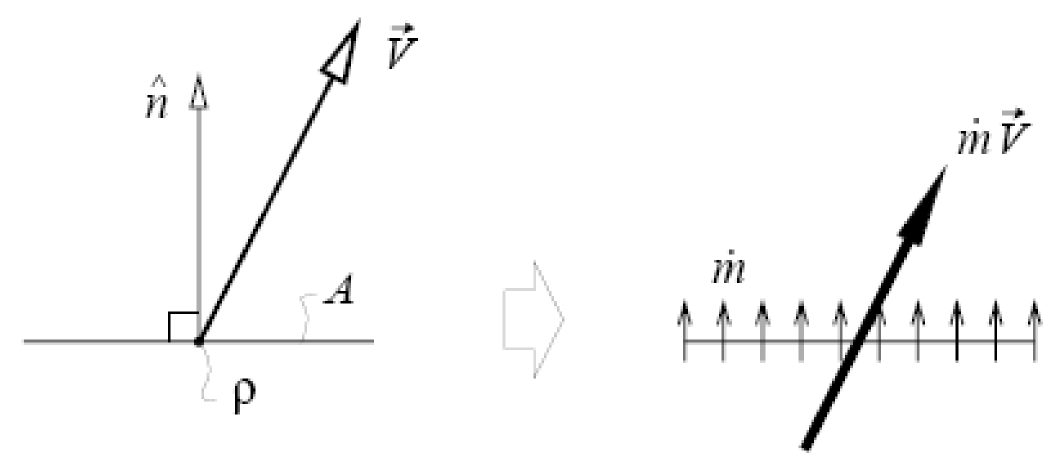


< 2.5. Momentum equations >

❖ Momentum flow

- Note that while mass flow is a scalar, the *momentum flow is a vector*, and points in the same direction as ***V***. The momentum flux vector is defined simply as the momentum flow per area.

$$\overrightarrow{momentum\ flow} = \rho V_n \vec{V}$$



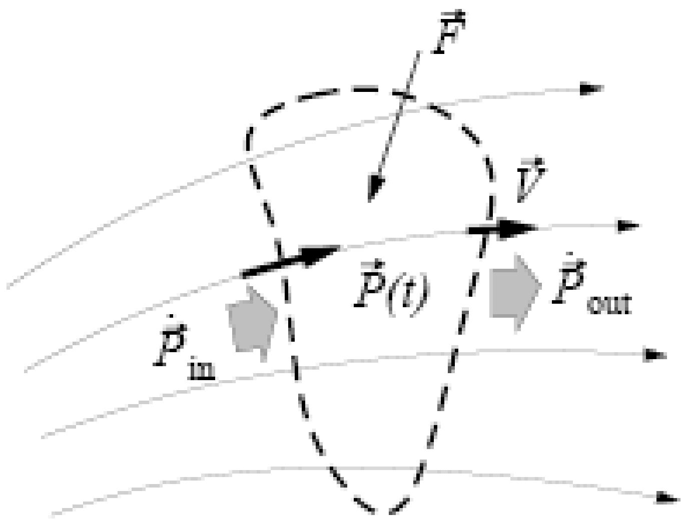
< 2.5. Momentum equations >

❖ Momentum conservation

- Newton's second law states that during a short time interval *dt*, the impulse of a force ***F*** applied to some affected mass, will produce a momentum change $d\boldsymbol{P}_a$ in that affected mass. When applied to a fixed control volume, this principle becomes

$$\frac{d\vec{P}_a}{dt} = \vec{F} \qquad (1)$$

$$\frac{d\vec{P}}{dt} + \dot{\vec{P}}_{out} - \dot{\vec{P}}_{in} = \vec{F} \qquad (2)$$

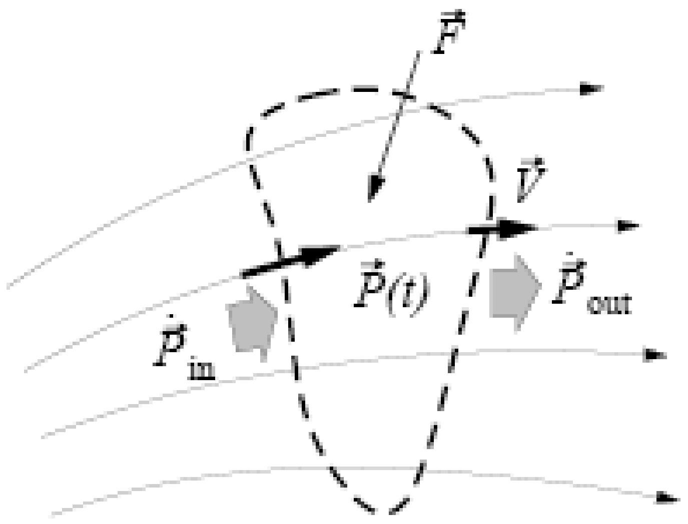


< 2.5. Momentum equations >

❖ Momentum conservation

- In the second equation (2), ***P*** is defined as the instantaneous momentum inside the control volume.

$$\vec{P}(t) \equiv \iiint \rho \vec{V} dv$$

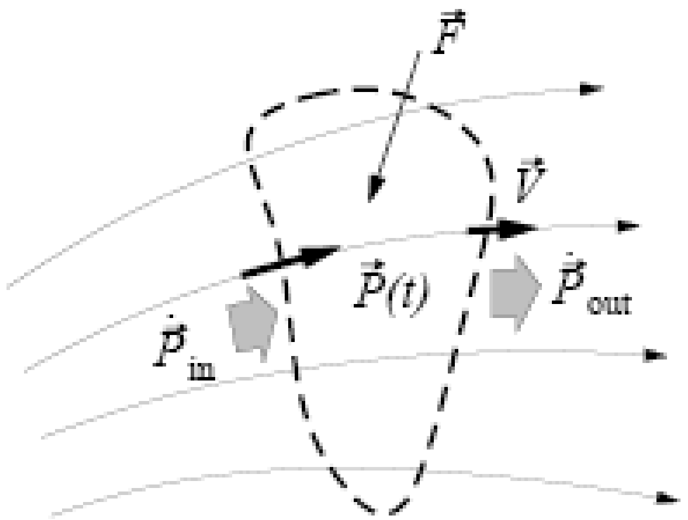


< 2.5. Momentum equations >

❖ Momentum conservation

- The $\boldsymbol{P}_{out}$ is added because mass leaving the control volume carries away momentum provided by $\boldsymbol{F}$, which $\boldsymbol{P}$ alone doesn't account for.

- The $\boldsymbol{P}_{in}$ is subtracted because mass flowing into the control volume is incorrectly accounted in $\boldsymbol{P}$, and hence must be discounted.



< 2.5. Momentum equations >

❖ Momentum conservation

- Both terms are evaluated by a surface integral of the momentum flux over the entire boundary.

$$\dot{\vec{P}}_{out} - \dot{\vec{P}}_{in} = \oiint \rho\left(\vec{V} \cdot \hat{n}\right)\vec{V} dA$$

- The sign of ***V·n*** automatically accounts for both inflow and outflow.

< 2.5. Momentum equations >

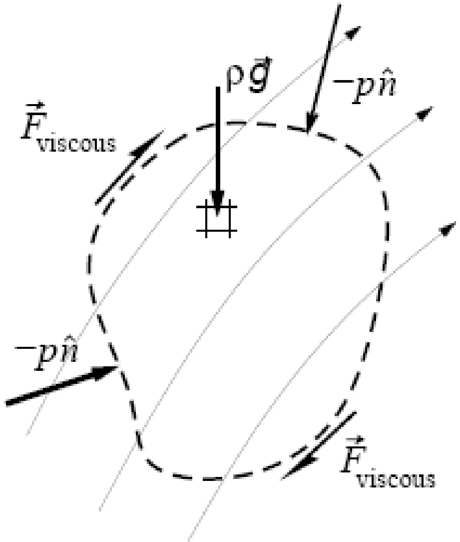
❖ Applied forces

- The force ***F*** consists of three types.

  - *Body forces* :

    These act on fluid inside the volume. The most common example is the gravity force, along the gravitational acceleration vector ***g***.

$$\vec{F}_{gravity} = \iiint \rho \vec{g} \, dv$$



< 2.5. Momentum equations >

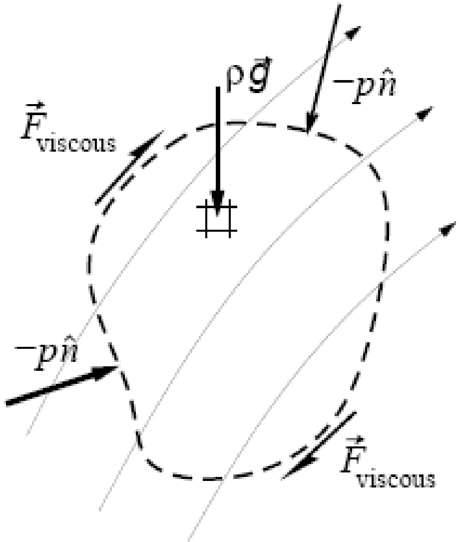
❖ Applied forces

- The force ***F*** consists of three types. (cont'd)

  - *Surface forces* :

    These act on the surface of the volume, and can be separated into pressure and viscous forces.

$$\vec{F}_{pressure} = \oiint - p\hat{n}\, dA$$

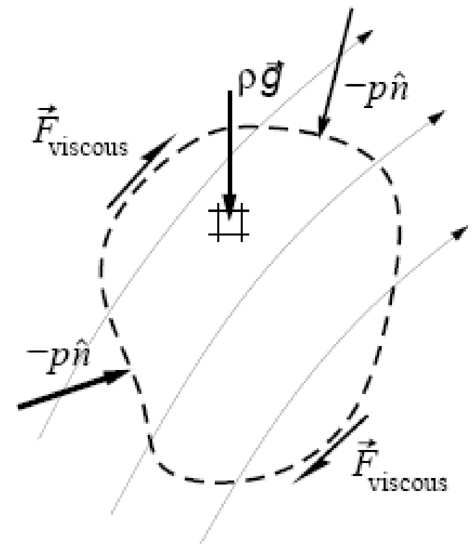


< 2.5. Momentum equations >

❖ Applied forces

- The force ***F*** consists of three types. (cont'd)
  - *Viscous forces* :

    The viscous force is complicated to write out, and for now will simply be called $\boldsymbol{F}_{viscous}$.



< 2.5. Momentum equations >

❖ Integral momentum equation

- Substituting all the momentum, momentum flow, and force definitions into Newton's second law (2) gives the *Integral Momentum Equation*.

$$\frac{d}{dt}\iiint \rho\vec{V}dv + \oiint \rho\left(\vec{V}\cdot\hat{n}\right)\vec{V}dA = \oiint - p\hat{n}dA + \iiint \rho\vec{g}dv + \vec{F}_{viscous} \quad (3)$$

- Along with the *Integral Mass Equation*, this equation can be applied to solve many problems involving finite control volumes.

< 2.5. Momentum equations >

❖ Differential momentum equation

- The pressure surface integral in equation (3) can be converted to a volume integral using the Gradient Theorem.

$$\oiint p\hat{n}dA = \iiint \nabla p dv$$

< 2.5. Momentum equations >

❖ Differential momentum equation

- The momentum-flow surface integral is also similarly converted using Gauss's Theorem. This integral is a vector quantity, and for clarity the conversion is best done on each component separately.

- After substituting ***V***=*u**i***+*v**j***+*w**k***, we have

$$\oiint \rho\left(\vec{V}\cdot\hat{n}\right)\left(u\hat{i}+v\hat{j}+w\hat{k}\right)dA=\hat{i}\iiint\nabla\cdot\left(\rho\vec{V}u\right)dv$$
$$+\hat{j}\iiint\nabla\cdot\left(\rho\vec{V}v\right)dv+\hat{k}\iiint\nabla\cdot\left(\rho\vec{V}w\right)dv$$

< 2.5. Momentum equations >

Differential momentum equation

- The x-component of the integral momentum equation (3) can now be written strictly in terms of volume integrals.

$$\iiint\left[\frac{\partial(\rho u)}{\partial t}+\nabla\cdot\left(\rho u\vec{V}\right)+\frac{\partial p}{\partial x}-\rho g_x-\left(F_x\right)_{viscous}\right]dv=0 \qquad (4)$$

This relation must hold for any control volume whatsoever.

< 2.5. Momentum equations >

❖ Differential momentum equation

- If we place an infinitesimal control volume at every point in the flow and apply equation (4), we can see that the whole quantity in the brackets must be zero at every point. This results in the x-Momentum Equation

$$\frac{\partial(\rho u)}{\partial t}+\nabla\cdot\left(\rho u\vec{V}\right)=-\frac{\partial p}{\partial x}+\rho g_x+\left(F_x\right)_{viscous} \qquad (5)$$

< 2.5. Momentum equations >

❖ Differential momentum equation

- The *y*- and *z*-Momentum Equations follow by the same process.

$$\frac{\partial(\rho v)}{\partial t}+\nabla\cdot\left(\rho v\vec{V}\right)=-\frac{\partial p}{\partial y}+\rho g_y+\left(F_y\right)_{viscous} \qquad (6)$$

$$\frac{\partial(\rho w)}{\partial t}+\nabla\cdot\left(\rho w\vec{V}\right)=-\frac{\partial p}{\partial z}+\rho g_z+\left(F_z\right)_{viscous} \qquad (7)$$

- These three equations are the embodiment of the Newton's second law of motion, applied at every point in the flow-field. The steady flow version has the $\partial/\partial t$ terms omitted.